Exercice sheet 1

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Exercice 1

This is an alternative solution to exercise 1 part a) from problem sheet 1. The official solution is available on the course website.

Recall that the σ-algebra generated by a random variable $X : \Omega \to \mathbb{R}$ is the σ-algebra generated by the collection of sets of the form $X^{-1}(B) := \{X \in B\}$, for B in the Borel sigma algebra $\mathcal{B}(\mathbb{R})$. In other words the σ-algebra generated by the random variable X is the smallest σ-algebra containing the sets of the form $X^{-1}(B)$, for B in the Borel sigma algebra $\mathcal{B}(\mathbb{R})$.

To solve the exercise, we will first calculate the sets of the form $\{X \in B\}$, for B in the Borel sigma algebra $\mathcal{B}(\mathbb{R})$ and then find the smallest σ-algebra containing those sets.

- $\mathcal{F}_0 := \sigma(X_0)$
  We have assumed that $X_0 = 8$ is constant, i.e. $X_0(\omega) = 8\ \forall \omega \in \Omega$. We therefore have that

$$X_0^{-1}(B) := \{X_0 \in B\} := \{\omega \in \Omega | X_0(\omega) \in B\} = \begin{cases} \Omega, & \text{if } 8 \in B \\ \emptyset, & \text{otherwise} \end{cases}$$

  The smallest σ-algebra containing $\emptyset$ and Ω is $\{\emptyset, \Omega\}$ since it is easy the verify that $\{\emptyset, \Omega\}$ is a σ-algebra (just need to show the three properties defining a σ-algebra).
  Hence $\mathcal{F}_0 := \sigma(X_0) = \{\emptyset, \Omega\}$

- $\mathcal{G}_1 := \sigma(X_1)$
  By definition $X_1(\omega) = X_0(\omega)Y_1(\omega) = 8Y_1(\omega)$. Using that the random variable Y_1 takes the following values:

$$Y_1(\omega) := \begin{cases} 2, & \text{if } \omega \in \{UU, UD\} \\ \frac{1}{2}, & \text{if } \omega \in \{DD, DU\} \end{cases}$$

  we get that the random variable X_1 can take the following values:

$$X_1(\omega) := \begin{cases} 8 * 2 = 16, & \text{if } \omega \in \{UU, UD\} \\ 8 * \frac{1}{2} = 4, & \text{if } \omega \in \{DD, DU\} \end{cases}$$

  The sets of the form $X_1^{-1}(B)$, for B in the Borel sigma algebra $\mathcal{B}(\mathbb{R})$ are thus:

$$X_1^{-1}(B) := \{X_1 \in B\} := \{\omega \in \Omega | X_1(\omega) \in B\} = \begin{cases} \Omega, & \text{if } 4 \in B \text{ and } 16 \in B \\ \{UU, UD\}, & \text{if } 16 \in B \text{ and } 4 \notin B \\ \{DD, DU\}, & \text{if } 4 \in B \text{ and } 16 \notin B \\ \emptyset, & \text{if } 4 \notin B \text{ and } 4 \notin B \end{cases}$$

  Again it is easy to check that $\{\emptyset, \Omega, \{UU, UD\}, \{DD, DU\}\}$ is a σ algebra and hence the smallest σ algebra containing the sets of the form $X_1^{-1}(B)$ is $\{\emptyset, \Omega, \{UU, UD\}, \{DD, DU\}\}$. Hence $\mathcal{G}_1 := \sigma(X_1) = \{\emptyset, \Omega, \{UU, UD\}, \{DD, DU\}\}$

- $\mathcal{F}_1 := \sigma(X_0, X_1)$
  We have calculated above that

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$$X_0^{-1}(B) := \{X_0 \in B\} := \{\omega \in \Omega | X_0(\omega) \in B\} = \begin{cases} \Omega, & \text{if } 8 \in B \\ \emptyset, & \text{otherwise} \end{cases}$$

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$$X_1^{-1}(B) := \{X_1 \in B\} := \{\omega \in \Omega | X_1(\omega) \in B\} = \begin{cases} \Omega, & \text{if } 4 \in B \text{ and } 16 \in B \\ \{UU, UD\}, & \text{if } 16 \in B \text{ and } 4 \notin B \\ \{DD, DU\}, & \text{if } 4 \in B \text{ and } 16 \notin B \\ \emptyset, & \text{if } 4 \notin B \text{ and } 4 \notin B \end{cases}$$

Morever we have seen that $\{\emptyset, \Omega, \{UU, UD\}, \{DD, DU\}\}$ is a σ-algebra. Hence the smallest σ algebra containing the sets of the form $X_0^{-1}(B)$ and $X_1^{-1}(B)$ is $\{\emptyset, \Omega, \{UU, UD\}, \{DD, DU\}\}$. Hence $\mathcal{F}_1 := \sigma(X_0, X_1) = \{\emptyset, \Omega, \{UU, UD\}, \{DD, DU\}\}$

A similar argument can be used to find $\mathcal{F}_2$ and $\mathcal{G}_2$. This is left as an exercise.